arXiv:1211.5051v1 [gr-qc] 21 Nov 2012

On the super-energy radiative gravitational fields

Joan Josep Ferrando[1] **and Juan Antonio Sáez**[2]

[1] Departament d'Astronomia i Astrofísica, Universitat de València, E-46100 Burjassot, València, Spain.

[2] Departament de Matemàtiques per a l'Economia i l'Empresa, Universitat de València, E-46071 València, Spain

E-mail: `joan.ferrando@uv.es; juan.a.saez@uv.es`

Abstract. We extend our recent analysis (*Class. Quantum Grav.* **29** 075012) on the Bel radiative gravitational fields to the super-energy radiative gravitational fields defined by García-Parrado (*Class. Quantum Grav.* **25** 015006). We give an intrinsic characterization of the new radiative fields, and we consider some distinguished classes of both radiative and non radiative fields. Several super-energy inequalities are improved.

PACS numbers: 04.20.C, 04.20.-q

1. Introduction

Elsewhere [1] we have analyzed the Bel concept of intrinsic radiative gravitational field [2, 3, 4], and we have shown that the three radiative types, N, III and II, correspond with three different physical situations: pure radiation, asymptotic pure radiation and generic (non-pure, non-asymptotic pure) radiation. In the aforementioned paper we have also shown that, for Bel non radiative fields, the minimum value of the relative super-energy is acquired by the observers at rest with respect to the field (those seeing a vanishing super-Poynting vector).

Following Bel's ideas, García-Parrado [5] has introduced new relative super-energy quantities in a recent paper and writes the full set of equations for these super-energy quantities. This study leads naturally to a concept of intrinsic radiation which is less restrictive than that given by Bel. Here we extend our analysis [1] on the Bel approach to the García-Parrado radiative gravitational fields.

In section 2 we introduce the basic concepts and notation and summarize previous results which help us to understand the present paper. The super-energy inequalities presented in [1] are revisited in section 3, where we extend the kind of bounderies already known for the super-energy density to the tensorial spacetime relative quantities.

In section 4 we define the *proper amount* of a super-energy quantity and show that, for Bel non radiative fields, it is acquired by the observers at rest and, for Bel radiative

fields, it is the asymptotic limit of the amount of this relative super-energy quantity. We also define the principal super-stresses of a Bel non radiative field.

Section 5 is devoted to studying both the García-Parrado radiative and non-radiative gravitational fields. We show that the non radiative fields correspond to type D metrics and some classes of type I metrics already considered in the literature: the IM^+ and the IM^∞ metrics [6]. These classes appear in a natural way when classifying the Bel-Robinson tensor as an endomorphism [7, 8]. Moreover, in these spacetimes the four null Debever directions span a 3-plane [6, 9, 10]. On the other hand, we show that three classes of García-Parrado type I radiative fields can be considered: the IM^-, the IM^{-6} and the generic type I_r metrics. The relationships between these classes and the Debever directions and with algebraic properties of the Bel-Robinson tensor are also outlined. Both, radiative and non radiative classes are characterized in terms of the principal super-stresses.

In section 6 we analyze the results of the paper with several diagrams which clarify the relation between the different classes of type I fields, and we describe ongoing work in studying the radiative character of a solution.

Finally, we present four appendices that make the work self-contained. The first one explains some notation. The second one summarizes the algebraic characterization and the canonical forms of the Bel-Robinson tensor. The third one presents some constraints on the amount of the super-energy quantities. Finally, the fourth one gives accurate proof of propositions 3 and 4.

2. The Bel approach to radiative gravitational states

With the purpose of defining intrinsic states of gravitational radiation, Bel [2, 3, 4] introduced the *super-energy Bel tensor* which plays an analogous role for gravitation to that played by the Maxwell-Minkowski tensor for electromagnetism. In the vacuum case this *super-energy Bel tensor* is divergence-free and it coincides with the *super-energy Bel-Robinson tensor* T.

Using tensor T, Bel defined the relative *super-energy density* and the *super-Poynting vector* associated with an observer. Then, following the analogy with electromagnetism, the intrinsic radiative gravitational fields are those for which the Poynting vector does not vanish for any observer [2, 4].

This analogy with electromagnetism also plays a fundamental role in our analysis [1] of the Bel radiative and non-radiative fields. Now we summarize our main results introducing the basic concepts required in the present work.

2.1. The Bel-Robinson tensor. Algebraic restrictions

In terms of the Weyl tensor W, the Bel-Robinson tensor takes the expression [2, 3, 4]:

$$T_{\alpha\mu\beta\nu} = \frac{1}{4}\left(W_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} W_{\mu\rho\nu\sigma} + *W_{\alpha}{}^{\rho}{}_{\beta}{}^{\sigma} *W_{\mu\rho\nu\sigma}\right) , \tag{1}$$

For any observer u, the relative *electric* and *magnetic* Weyl fields are given by $E = W(u;u)$ and $H = *W(u;u)$, respectively. The following relative super-energy quantities can be defined:

$$\tau = T(u,u,u,u) , \quad q_\perp = -T(u,u,u)_\perp , \quad t_\perp = T(u,u)_\perp , \quad Q_\perp = -T(u)_\perp , \quad T_\perp , \tag{2}$$

where, for a tensor A, $A_\perp$ denotes the orthogonal projection defined by the projector $\gamma = u \otimes u + g$.

Bel introduced the *super-energy density* τ and the *super-Poynting (energy flux) vector* $q_\perp$ years ago [2, 4]. Bonilla and Senovilla [11] used $t_\perp$ in studying the causal propagation of gravity and, recently, García-Parrado [5] has considered $Q_\perp$ and $T_\perp$. These last three relative quantities have been called the *super-stress tensor*, the *stress flux tensor* and the *stress-stress tensor*, respectively.

In vacuum, the Bianchi identities imply that T satisfies the conservation equation $\nabla \cdot T = 0$. For any observer, this equation shows that the relative quantities $q_\perp$ and $Q_\perp$ play the role of fluxes of the relative quantities τ and $t_\perp$, respectively [5].

The algebraic constraints on the Bel-Robinson tensor playing a similar role to that played by the Rainich conditions [12] for the electromagnetic energy tensor were obtained by Bergqvist and Lankinen [13].

On the other hand, we have studied elsewhere [7, 8] the Bel-Robinson tensor T as an endomorphism on the 9-dimensional space of the traceless symmetric tensors. Its nine eigenvalues $\{t_k, \tau_k, \bar{\tau}_k\}$ depend on the three complex Weyl eigenvalues $\{\rho_k\}$ as

$$t_k = |\rho_k|^2; \qquad \tau_k = \rho_i \bar{\rho}_j, \quad (ijk) \equiv \textit{ even permutation of } (123) . \tag{3}$$

We have also intrinsically characterized the algebraic classes of T [7], and we have given their Segrè type and their canonical form [8]. Some of these results which we need here are summarized in Appendix B.

2.2. Bel radiative gravitational fields

The super-energy density τ vanishes only when the Weyl tensor W vanishes. Then, if we consider τ as a measure of the gravitational field, its flux $q_\perp$ denotes the presence of gravitational radiation. This is the point of view of Bel [2, 4], who gave the following definition.

Definition 1 (Intrinsic gravitational radiation, Bel 1958) *In a vacuum spacetime there exists* intrinsic gravitational radiation *(at a point) if the super-Poynting vector $q_\perp$ does not vanish for any observer.*

It is known that the Bel radiative gravitational fields are those of Petrov-Bel types N, III and II [4]. Then, also motivated by the Lichnerowicz ideas [14], we have proposed to distinguish three physical situations [1]:

Definition 2 *In a vacuum spacetime there exists* pure gravitational radiation *(at a point) when the whole super-energy density is radiated as Poynting super-energy,* $\tau = |q_\perp|$.

In a vacuum spacetime there exists asymptotic pure gravitational radiation *(at a point) when* $\tau \neq |q_\perp|$ *and for any positive real number* ϵ *one can find an observer for which the non radiated energy* $\tau - |q_\perp|$ *is smaller than* ϵ.

The non pure, non asymptotic pure radiative gravitational fields are named generic radiative states.

Proposition 1 *The pure radiative states are the type* N *gravitational fields.*

The asymptotic pure radiative states are the type III *gravitational fields.*

The generic radiative states are the type II *gravitational fields.*

2.3. Bel non radiative gravitational fields. Observer at rest and proper super-energy density

From Bel's point of view, non radiative gravitational fields are those for which at least one observer exists who sees a vanishing relative super-Poynting vector. The following definition naturally arises:

Definition 3 *The observers for whom the super-Poynting vector vanishes, are said to be* observers at rest *with respect to the gravitational field.*

Then, we have the following significant and known result [4]:

Proposition 2 *The non radiative gravitational fields are the Petrov-Bel type* I *or* D *spacetimes.*

The observers at rest with respect to the gravitational field are those for whom the electric and magnetic Weyl tensors simultaneously diagonalize.

In a type I *spacetime a unique observer* e_0 *at rest with respect the gravitational field exists.*

In a type D *spacetime the observers* e_0 *at rest with respect to the gravitational field are those lying on the Weyl principal plane.*

In [1] we have given the following definition and result:

Definition 4 *We call* proper super-energy density *of a gravitational field the invariant scalar*

$$\xi \equiv \frac{1}{4}\sum_{i=1}^{3} t_i \, . \tag{4}$$

Theorem 1 *For a Bel non radiative gravitational field (I or D) the minimum value of the relative super-energy density is the proper super-energy density ξ, which is acquired by the observers at rest with respect to the field.*

Let ℓ be the fundamental direction of a Bel radiative gravitational field (N, III or II). For a family of observers having each a spatial velocity at a point tangent and parallel to $\ell_\perp$, the super-energy density measured by an observer at the same point decreases and tends to the proper super-energy density ξ as its spatial velocity increase and approaches the speed of light.

For pure and asymptotic pure radiation (N or III), the proper super-energy density ξ is zero. For generic radiation (type II), ξ is strictly positive.

3. Super-energy inequalities

The Bel-Robinson tensor satisfies the generalized dominant energy condition [15, 16] which implies that for any observer u, the relative quantities τ and $q = -T(u, u, u)$ are submitted to the known inequalities, $\tau \geq 0$, $(q, q) \leq 0$. In [1] we have generalized these *super-energy inequalities* in two aspects. On the one hand we have shown that τ and (q, q) are bounded by scalars depending on the main quadratic invariant

$$\alpha \equiv \frac{1}{2}\sqrt{(T, T)}\,, \quad (T, T) = T_{\alpha\beta\lambda\mu}T^{\alpha\beta\lambda\mu} = \frac{1}{64}[(W, W)^2 + (W, *W)^2] \geq 0\,. \quad (5)$$

On the other hand we have extended these kind of boundaries to other spacetime relative quantities (see theorem 2 in [1]). Now we present stronger inequalities on these spacetime quantities, restrictions that lead naturally to the concept of proper amount of the super-energy quantities. We will use the following quadratic scalar invariant:

$$\chi \equiv \frac{1}{4}\sum_{i=1}^{3} t_i^2\,, \qquad 8\xi^2 + \alpha^2 = 6\chi\,, \qquad (6)$$

where the above constraint between the invariants α, χ and ξ is a consequence of the restrictions on the Bel-Robinson eigenvalues given in Appendix B.

In Appendix D we will prove the following propositions.

Proposition 3 *Let α and χ be the Bel-Robinson invariants defined in (5) and (6), and $t = T(u, u)$ for any observer u. Then, it holds:*

- *For type N,* $(t, t) = \chi = \frac{1}{2}\alpha^2 = 0$.
- *For type III,* $(t, t) > \chi = \frac{1}{2}\alpha^2 = 0$.
- *For type II,* $(t, t) > \chi = \frac{1}{2}\alpha^2 > 0$.
- *For type D,* $(t, t) \geq \chi = \frac{1}{2}\alpha^2 > 0$.
- *For type I,* $(t, t) \geq \chi \geq \frac{1}{2}\alpha^2 \geq 0$.

Moreover: (i) for types I and D, $(t_0, t_0) = \chi$, with $t_0 = T(e_0, e_0)$, e_0 being a principal observer, and (ii) for types III and II, (t, t) tends to χ as the spatial velocity of the observer approaches the speed of light in the direction $\ell_\perp$, ℓ being the fundamental direction of the field.

Proposition 4 *Let ξ and α be the Bel-Robinson invariants defined in (4) and (5), and $q = -T(u, u, u)$ for any observer u. Then, it holds:*

- *For type N,* $(q, q) = -\xi^2 = -\frac{1}{4}\alpha^2 = 0$.
- *For type III,* $(q, q) < -\xi^2 = -\frac{1}{4}\alpha^2 = 0$.
- *For type II,* $(q, q) < -\xi^2 = -\frac{1}{4}\alpha^2 < 0$.
- *For type D,* $(q, q) \leq -\xi^2 = -\frac{1}{4}\alpha^2 < 0$.
- *For type I,* $(q, q) \leq -\xi^2 \leq -\frac{1}{4}\alpha^2 \leq 0$.

Moreover: (i) for types I and D, $(q_0, q_0) = -\xi^2$*, with* $q_0 = -T(e_0, e_0, e_0)$*,* e_0 *being a principal observer, and (ii) for types III and II,* (q, q) *tends to* $-\xi^2$ *as the spatial velocity of the observer approaches the speed of light in the direction* $\ell_\perp$*,* ℓ *being the fundamental direction of the field.*

Now we can state the following theorem.

Theorem 2 (Super-energy inequalities) *Let T be the Bel-Robinson tensor and for any observer u let us define the relative spacetime quantities:*

$$Q = -T(u)\,, \quad t = T(u, u)\,, \quad q = -T(u, u, u)\,, \quad \tau = T(u, u, u, u)\,. \tag{7}$$

Then, the following super-energy inequalities hold:

$$\begin{gathered}(T, T) \equiv 4\alpha^2 \geq 0\,, \qquad (Q, Q) = -\alpha^2 \leq 0\,, \\ (t, t) \geq \chi \geq \tfrac{1}{2}\alpha^2 \geq 0\,, \quad (q, q) \leq -\xi^2 \leq -\tfrac{1}{4}\alpha^2 \leq 0\,, \quad \tau \geq \xi \geq \tfrac{1}{2}\alpha \geq 0\,.\end{gathered} \tag{8}$$

where ξ and χ are the invariant scalars defined in (4) and (6).

The first, second and last conditions in (8) have been stated and shown in [1] (theorems 1 and 2). The third condition in (8) is a consequence of proposition 3 and, finally, the fourth condition in (8) is a consequence of proposition 4.

4. Lower bounds on the amount and proper amount of the super-energy quantities

For any observer u, the amount of the relative super-energy quantities (2) are the relative scalars given by

$$\tau\,, \quad |q_\perp| = \sqrt{(q_\perp, q_\perp)}\,, \quad |t_\perp| = \sqrt{(t_\perp, t_\perp)}\,, \quad |Q_\perp| = \sqrt{(Q_\perp, Q_\perp)}\,, \quad |T_\perp| = \sqrt{(T_\perp, T_\perp)}\,.$$

Theorem 1 shows that the proper super-energy density ξ is the infimum, for all the observers u, of the super-energy densities τ_u. This property justifies the following definition.

Definition 5 *We call* proper amount of a super-energy quantity *the infimum of the amounts with respect all the observers of this quantity.*

The two last inequalities in (8) imply:

$$\tau^2 \geq \tau^2 - |q_\perp|^2 \geq \xi^2 \,, \qquad |q_\perp| \geq 0 \,. \tag{9}$$

Moreover, as a consequence of proposition 4, equalities in (9) hold for the observers at rest in the case of Bel non radiative fields and, for Bel radiative fields, the inequalities (9) approach an equality as the spatial velocity of the observer approaches the speed of light in direction $\ell_\perp$. On the other hand, from (9) and the quadratic scalar constraints (C.3) we obtain:

$$\begin{aligned}
|t_\perp|^2 &= \frac{1}{3}\,\tau^2 + \frac{1}{6}\,\alpha^2 + \frac{2}{3}\,|q_\perp|^2 \geq\ \frac{1}{3}\xi^2 + \frac{1}{6}\alpha^2 \,, \\
|Q_\perp|^2 &= 2\,\tau^2 - \frac{1}{2}\,\alpha^2 -\ |q_\perp|^2 \geq\ 2\xi^2 - \frac{1}{2}\alpha^2 \,, \\
|T_\perp|^2 &= 5\,\tau^2 + \alpha^2 - 4\,|q_\perp|^2 \geq\ 5\xi^2 + \alpha^2 \,.
\end{aligned} \tag{10}$$

Moreover, equalities in (10) hold for the observers at rest in the case of Bel non radiative fields and, for Bel radiative fields, the inequalities (10) approach an equality as the spatial velocity of the observer approaches the speed of light in direction $\ell_\perp$. Consequently, we can state the following theorem.

Theorem 3 *The proper amount of radiated super-energy ξ_q, of super-stress ξ_t, of radiated super-stress ξ_Q and stress-stress ξ_T are, respectively, the invariant scalars:*

$$\xi_q = 0 \,, \qquad \xi_t = \sqrt{\frac{1}{3}\xi^2 + \frac{1}{6}\alpha^2} \,, \qquad \xi_Q = \sqrt{2\xi^2 - \frac{1}{2}\alpha^2} \,, \qquad \xi_T = \sqrt{5\xi^2 + \alpha^2} \,.$$

where ξ and α are the proper energy density (4) and the main quadratic scalar (5).

For a Bel non radiative gravitational field (I or D) the proper amount of each super-energy quantity is acquired by the observers at rest with respect to the field: if e_0 is such an observer, then

$$\tau_0 = \xi \,, \qquad |q_{0\perp}| = \xi_q = 0 \,, \qquad |t_{0\perp}| = \xi_t \,, \qquad |Q_{0\perp}| = \xi_Q \,, \qquad |T_{0\perp}| = \xi_T \,.$$

Let ℓ be the fundamental direction of a Bel radiative gravitational field (N, III or II). For a family of observers having each a spatial velocity at a point tangent and parallel to $\ell_\perp$, the amount of each super-energy quantity measured by an observer at the same point tends to the proper amount of this quantity as its spatial velocity increase and approaches the speed of light.

On the other hand, from expressions (10) we obtain a result which extends a previous one given in [11]:

Proposition 5 *The amount of the super-energy quantities are submitted to the following constraints:*

$$|Q_\perp|^2 - |q_\perp|^2 = 3(\tau^2 - |t_\perp|^2) \geq 0 \,, \qquad 3\tau - |T_\perp| \geq 0 \,. \tag{11}$$

Note that for Bel non radiative fields (types I and D) we can consider the super-energy quantities relative to the observers e_0 at rest with respect to the field: the *proper super-energy* $\tau_0 = \xi$, the *proper Poynting vector* $q_{0\perp} = 0$, the *proper super-stress tensor* $t_{0\perp}$, the *proper super-stress flux tensor* $Q_{0\perp}$ and the *proper stress-stress tensor* $T_{0\perp}$. If $\{e_0, e_i\}$ is a Weyl canonical frame of a type I or type D spacetime, from the canonical form B.6 we obtain:

$$t_{0\perp} = \sum_{i=1}^{3} \kappa_i \, e_i \otimes e_i \, , \qquad 4\kappa_i = t_k + t_j - t_i \, , \quad i \neq j \neq k \neq i \, , \tag{12}$$

$$Q_{0\perp} = \kappa \sum_{\sigma} (e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}) \, , \qquad \kappa^2 = \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 \, , \tag{13}$$

where σ is a permutation of (123).

The three eigenvalues κ_i of the super-stress tensor $t_{0\perp}$ play for gravitation the same role played by the principal stresses for electromagnetism. Then, they are the *principal super-stresses*, which are submitted to the constraint

$$\kappa_1 + \kappa_2 + \kappa_3 = \xi \, . \tag{14}$$

It is worth remarking that in type D the proper super-stress tensor $t_{0\perp}$ depends on the chosen principal observer. Nevertheless, the principal super-stresses do not.

5. García-Parrado radiative gravitational fields

The super-stress tensor $t_\perp$ vanishes only when the Weyl tensor W vanishes. Thus we can also consider $t_\perp$ as a measure of the gravitational field. Then its flux $Q_\perp$ denotes the presence of gravitational radiation. This fact has been pointed out by García-Parrado [5], who has given the following definition.

Definition 6 (Intrinsic super-energy radiation, García-Parrado 2008) *In a vacuum spacetime there exists* intrinsic super-energy radiation *(at a point) if the stress flux tensor $Q_\perp$ does not vanish for any observer.*

For any observer we have $\mathrm{tr}\, Q_\perp = q_\perp$. Then, $q_\perp$ vanishes when $Q_\perp$ vanishes. Consequently,

Proposition 6 *Every Bel radiative gravitational field is a García-Parrado radiative gravitational field.*

Every García-Parrado non-radiative gravitational field is a Bel non-radiative gravitational field, and the observers who do not see stress flux ($Q_\perp = 0$) are the observers at rest with respect to the field.

Thus, the definition given by García-Parrado is less restrictive than Bel's definition, and it allows type I radiative gravitational fields [5]. Now we analyze them and consider several significant classes of both radiative and non-radiative García-Parrado gravitational fields.

5.1. Super-energy non radiative gravitational fields

After the above proposition, the García-Parrado non radiative fields are type I or type D metrics with vanishing proper stress flux tensor $Q_{0\perp}$, or equivalently, with vanishing proper amount ξ_Q. This condition is equivalent to the Bel-Robinson tensor having real eigenvalues as a consequence of expressions (13) and (B.1). On the other hand García-Parrado [5] showed that $Q_{0\perp} = 0$ if, and only if, the proper electric and magnetic Weyl tensors are linearly dependent. Thus we can state.

Theorem 4 *The García-Parrado non radiative gravitational fields are the type I or type D metrics which satisfy one of the following equivalent conditions:*

(i) The proper stress flux tensor vanishes, $Q_{0\perp} = 0$.

(ii) The proper electric and magnetic Weyl tensors are linearly dependent, $E_0 \otimes H_0 = H_0 \otimes E_0$.

(iii) The proper amount of the stress flux vanishes, $\xi_Q \equiv \sqrt{2\xi^2 - \frac{1}{2}\alpha^2} = 0$.

(iv) The Bel-Robinson tensor has real eigenvalues.

Note that the characterizations (i) and (ii) of the above theorem make reference to relative quantities, namely the stress flux tensor and the electric and magnetic parts of the Weyl tensor. Nevertheless, the conditions (iii) and (iv) are intrinsic in the Bel-Robinson tensor T and they impose respectively, the vanishing of an invariant scalar and an algebraic property of T.

As pointed out previously [5] all the type D gravitational fields satisfy conditions of theorem 4. Now we study the type I metrics which satisfy them.

Elsewhere [10] we have studied the aligned Weyl fields, i.e. the spacetimes with linearly dependent electric and magnetic Weyl tensors, and we have shown that they correspond to metrics of types IM^+ or IM^∞ in the classification of McIntosh-Arianrhod [6]. This means that the scalar invariant M defined in B.4 is, respectively, a positive real number or infinity, and they are the type I spacetimes where the four null Debever directions span a 3-plane [6, 9, 10]. Moreover, the (spatial) direction orthogonal to this 3-plane is the Weyl principal direction associated with the Weyl eigenvalue with the shortest modulus.

We can state this last condition in terms of the principal super-stresses κ_i. Indeed, say ρ_1 is the shortest Weyl eigenvalue, then t_1 is the shortest Bel-Robinson real eigenvalue and, from (12), κ_1 is the largest principal super-stress. Thus, we have:

Proposition 7 *A type I spacetime defines a super-energy non radiative gravitational fields if, and only if, the four null Debever directions span a 3-plane.*

Moreover, the direction orthogonal to this 3-plane is that defined by the eigenvector associated with the largest principal super-stress.

Finally, we analyze how to distinguish the three types of non radiative fields in terms of super-energy quantities. The results above imply that the all three cases,

IM^{+}, IM^{∞} and D, are submitted to the same constraints for the proper super-energy amounts:

$$\xi_q = \xi_Q = 0\,, \qquad \xi_T = 3\xi_t = 3\xi\,.$$

Nevertheless, we can distinguish these three types by using the principal super-stresses κ_i. In type D we have $t_2 = t_3 \neq 0$, $t_1 = 4t_2$, and then $\kappa_2 = \kappa_3 = -2\kappa_1 \neq 0$ as a consequence of (12). In type IM^{∞} we have $t_1 = t_2 \neq 0$, $t_3 = 0$, and then $\kappa_1 = \kappa_2 = 0$.

Proposition 8 *In a super-energy non radiative spacetime the principal super-stresses satisfy*

$$\kappa_1 \leq 0 \leq \kappa_2 \leq \kappa_3\,, \qquad |\kappa_1| \leq |\kappa_2|\,.$$

Moreover, the spacetime is:

Type D iff $\kappa_1 = -\frac{1}{2}\kappa_2 < 0 < \kappa_2 = \kappa_3$.
Type IM^{∞} iff $\kappa_1 = 0 = \kappa_2 < \kappa_3$.
Type IM^{+} otherwise, and then $\kappa_1 < 0 < \kappa_2 < \kappa_3$.

We can quote some examples of non radiative fields:

(i) All the type D vacuum solutions are known [17, 18].

(ii) A result inferred by Szekeres [19] and confirmed by Brans [20] states that no vacuum solutions of type IM^{∞} exist (see [22] for a generalization).

(iii) The purely electric or purely magnetic solutions are of type IM^{+} [23]. Every static solution has a purely electric Weyl tensor and, consequently, is of type IM^{+}. Both static and non static purely electric metrics can be found in the Kasner vacuum solutions [21, 18]. Some restriction are known on the existence of other IM^{+} vacuum solutions (see [10] and references therein).

5.2. Super-energy radiative gravitational fields

After propositions 6 and 7, the García-Parrado radiative fields are the Bel radiative fields (types N, III and II) and the type I metrics with non vanishing proper stress flux tensor $Q_{0\perp}$. After theorem 4, radiative type I fields can be characterized by the following equivalent conditions: (i) the proper electric and magnetic Weyl tensors are linearly independent, (ii) the proper amount of the stress flux does not vanish, (iii) the Bel-Robinson tensor has some complex eigenvalue, (iv) The four null Debever directions define a frame. Now we consider some relevant classes of type I radiative fields.

The invariant classification of the Bel-Robinson tensor [7, 8] leads to three classes with non real eigenvalues (see appendix): the most regular one I_r, and those with a double or a triple degeneration, which correspond to types IM^{-} or IM^{-6}, respectively, in the classification of McIntosh-Arianrhod [6]. Moreover, they are the type I spacetimes where the four null Debever directions define a frame. This frame is symmetric (four metrically indistinguishable vectors) for the case IM^{-6}, and a partially symmetric frame for IM^{-} [10]. Thus, we have:

Proposition 9 *A type I spacetime defines a super-energy radiative gravitational fields if, and only if, the four null Debever vectors define a frame.*

Moreover, this frame is symmetric (four metrically indistinguishable vectors) for type IM^{-6} spacetimes, and it is partially symmetric (two pairs of metrically indistinguishable vectors) for type IM^{-} spacetimes. Otherwise, the spacetime is of regular type I_r.

We can label the two degenerate classes in terms of the principal super-stresses κ_i. Indeed, from (12), condition $t_1 = t_2 = t_3$ implies three equal principal super-stresses, $\kappa_1 = \kappa_2 = \kappa_3$, and $t_1 \neq t_2 = t_3$ implies two equal principal super-stresses, $\kappa_1 \neq \kappa_2 = \kappa_3$.

Proposition 10 *A super-energy radiative type I spacetime is*

Type IM^{-6} iff $\kappa_1 = \kappa_2 = \kappa_3$.

Type IM^{-} iff $\kappa_1 = \kappa_2 \neq \kappa_3$.

Type I_r otherwise.

We can quote some examples of radiative fields:

(i) The Petrov homogeneous vacuum solution [24, 18] is of type IM^{-6}.

(ii) The windmill metric [25, 18] is a family of type IM^{-} vacuum solutions.

(iii) Vacuum solutions of generic radiative type I_r are quite abundant. We have, for example, the Taub family of metrics [26, 18] and its counterpart with timelike orbits, or its related windmill solutions [22].

6. An analysis of type I classes

The 'degenerate' type I classes defined in [6] in terms of the adimensional invariant M have a nice interpretation in terms of the null Debever directions [6, 9, 10]. Moreover, these classes also appear when classifying the Bel-Robinson tensor as an endomorphism on the nine-dimensional space of the trace-less symmetric tensors [7, 8].

In the present paper we have shown how Garcia-Parrado radiative and non radiative fields can be distinguished in terms of the invariant M. Figure 1(a) presents the complex plane where we can plot the different values of M. Out of the real axes, points correspond to generic radiative fields I_r. For real negative values of M we have the degenerate radiative fields IM^- and IM^{-6}. The real non negative values of M correspond to non radiative fields: type D when $M = 0$, type IM^+ when $M > 0$ and type IM^∞ when M is not defined.

The study of the different classes using the Bel-Robinson tensor [7, 8] implies analyzing the Bel-Robinson real eigenvalues t_i. And an approach using the super-energy relative magnituds can be built with the principal super-stresses κ_i.

6.1. A diagram approach using Bel-Robinson real eigenvalues

Constraints on the real eigenvalues t_i:

$$t_1 + t_2 + t_3 = 4\xi \,, \qquad t_1^2 + t_2^2 + t_3^2 = 4\chi \leq 8\xi^2 \,,$$

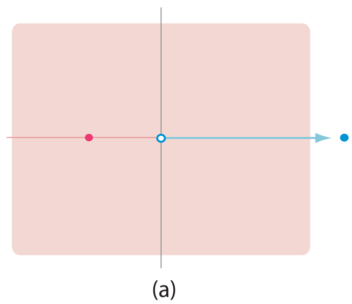

Figure 1. In these diagrams reddish points represent radiative metrics: red solid dot type IM^{-6}, red lines type IM^{-}, and type I_r otherwise. And bluish points represent non radiative metrics: blue solid dot type IM^{∞}, blue empty dot type D, and blue lines type IM^{+}. (a) Plane where the complex invariant M is plotted. (b) Plane $\bar{t}_1 + \bar{t}_2 + \bar{t}_3 = 1$ in the parameter space $\{\bar{t}_1, \bar{t}_2, \bar{t}_3\}$. (c) Plane $\bar{\kappa}_1 + \bar{\kappa}_2 + \bar{\kappa}_3 = 1$ in the parameter space $\{\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3\}$. In diagrams (b) an (c) the black triangle represents points on the corresponding coordinate planes.

Then, the adimensional parameters $\bar{t}_i = \dfrac{t_i}{4\xi}$ satisfy:

$$\bar{t}_1 + \bar{t}_2 + \bar{t}_3 = 1\,, \qquad \bar{t}_1^2 + \bar{t}_2^2 + \bar{t}_3^2 \leq \frac{1}{2}\,,$$

conditions which represent the points on a plane Π, and in a sphere S, respectively, in the parameter space $\{\bar{t}_1, \bar{t}_2, \bar{t}_3\}$. Every type I metric corresponds to a point on the circle surrounded by the intersection circumference $C = \Pi \cap S$. This C is the incircle of triangle T on Π defined by coordinate planes $\bar{t}_1\bar{t}_2\bar{t}_3 = 0$ (see Figure 1(b)). The non radiative fields are on circumference C, and the radiative fields are in its interior. The degenerate both non radiative and radiative fields are located on the medians of the triangle T: type IM^{∞} on the base points, type D on the three opposing points in C, type IM^{-6} on the barycenter and IM^{-} in any other point on the medians.

6.2. *A diagram approach using principal super-stress scalars*

Constraints on the principal super-stress scalars κ_i:

$$\kappa_1 + \kappa_2 + \kappa_3 = \xi\,, \qquad \kappa_1^2 + \kappa_2^2 + \kappa_3^2 = \xi^2 - 2\kappa^2 \leq \xi^2\,,$$

Then, the adimensional parameters $\bar{\kappa}_i = \dfrac{\kappa_i}{\xi}$ satisfy:

$$\bar{\kappa}_1 + \bar{\kappa}_2 + \bar{\kappa}_3 = 1\,, \qquad \bar{\kappa}_1^2 + \bar{\kappa}_2^2 + \bar{\kappa}_3^2 \leq 1\,,$$

conditions which represent the points on a plane Π, and in a sphere S, respectively, in the parameter space $\{\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3\}$. Every type I metric corresponds to a point on the circle surrounded by the intersection circumference $C = \Pi \cap S$. This C is the circumcircle of triangle T on Π defined by coordinate planes $\bar{\kappa}_1\bar{\kappa}_2\bar{\kappa}_3 = 0$ (see Figure 1(c)). The non radiative fields are on circumference C, and the radiative fields are in its interior. Both

non radiative and radiative degenerate fields are located on the medians of triangle T: type IM^∞ on the vertex points, type D on the three opposing points in C, type IM^{-6} on the barycenter and IM^- in any other point on the medians.

We can see in the diagram that non radiative types D and IM^∞ can be considered as the limit of the 'generic' non radiative type IM^+, but can also be the limit of radiative cases, in particular of the 'degenerate' radiative type IM^-. Type IM^{-6} is the farthest from the non radiative types, that is to say, it could be considered the most radiative type I case. This statement, based on geometric considerations, can also be supported by an analytical approach. Indeed, if we define the *adimensional radiation parameter* $\bar{\kappa}^2 = \frac{\kappa^2}{\xi^2}$, we have: $\bar{\kappa}^2$ vanishes for non radiative types and it reaches the maximum value for the radiative type IM^{-6}. The study of this or other similar radiation parameters for known vacuum solutions would be an interesting task which we will undertake elsewhere.

Acknowledgments

This work has been supported by the Spanish Ministerio de Ciencia e Innovación, MICIN-FEDER project FIS2009-07705.

Appendix A. Notation

(i) Composition of two 2-tensors A, B as endomorphisms: $A \cdot B$, $(A \cdot B)^\alpha{}_\beta = A^\alpha{}_\mu B^\mu{}_\beta$.

(ii) In general, for arbitrary tensors S, T, $S \cdot T$ will be used to indicate the contraction of adjacent indexes on the tensorial product. Similarly, $\overset{2}{\cdot}$ and $\overset{3}{\cdot}$ denote, respectively, a double and a triple contraction.

(iii) Square and trace of a 2-tensor A: $A^2 = A \cdot A, \quad \operatorname{tr} A = A^\alpha{}_\alpha$.

(iv) The action on one or more vectors of an arbitrary tensor S as multilinear form will be denoted $S(x)$, $S(x,y)$, $S(x,y,z)$,... For example, the action of a 2-tensor A as an endomorphism $A(x)$ and as a bilinear form $A(x,y)$:

$$A(x)^\alpha = A^\alpha{}_\beta x^\beta, \qquad A(x,y) = A_{\alpha\beta} x^\alpha y^\beta$$

(v) The quadratic scalar associated with an arbitrary tensor S will be denoted (S, S). For example, if x is a vector and A is a 2-tensor:

$$(x,x) = x^2 = g(x,x)\,, \qquad (A,A) = A^{\alpha\beta} A_{\alpha\beta}\,.$$

(vi) If S is a $p+q$-tensor and X is a q-tensor, then $S(X)$ denotes the p-tensor:

$$S(X)_{\underline{p}} = S_{\underline{p+q}} X^{\underline{q}}\,,$$

where underlining denotes multi-index.

Appendix B. Algebraic classification of the Bel-Robinson tensor

The Bel-Robinson tensor T defines an endomorphism on the space of the traceless symmetric 2-tensors [7, 8]. The nine eigenvalues $\{t_k, \tau_k, \bar{\tau}_k\}$ depend on the three (complex) Weyl eigenvalues $\{\rho_k\}$ as $t_k = |\rho_k|^2$, $\tau_k = \rho_i \bar{\rho}_j$, (ijk) being an even permutation of (123).

Three independent invariant scalars can be associated with T . In fact, the nine eigenvalues $\{t_i, \tau_i, \bar{\tau}_i\}$ can be written in terms of three scalars $\{\kappa_i\}$ as [7]:

$$t_i = 2(\kappa_j + \kappa_k), \quad \tau_i = -2(\kappa_i + \mathrm{i}\,\kappa), \quad \kappa^2 = \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 \,. \tag{B.1}$$

Note that scalars $\{\kappa_i\}$ satisfy the following inequalities:

$$\kappa_j + \kappa_k > 0, \quad (j \neq k)\,; \qquad \kappa^2 = \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 \geq 0 \,. \tag{B.2}$$

Conversely, the scalars κ_i depend on the three real Bel-Robinson eigenvalues $\{t_i\}$ as:

$$4\kappa_i = t_j + t_k - t_i, \qquad (i, j, k) \neq \tag{B.3}$$

The classification of T as an endomorphism [7, 8] leads to nine classes: the Petrov-Bel types O, N, III and II and five subclasses of type I metrics: types I_r, IM^-, IM^{-6}, IM^+ and IM^∞. These 'degenerate' type I metrics may be characterized in terms of the adimensional Weyl scalar invariant [6, 9]:

$$M = \frac{a^3}{b^2} - 6 \tag{B.4}$$

where a and b are the quadratic and the cubic Weyl complex scalar invariants. Now we summarize conditions which characterize every Bel-Robinson type and give the respective canonical form.

Type I_r

Let $\{e_0, e_i\}$ be the canonical frame of a type I Weyl tensor. We can define the traceless symmetric 2-tensors:

$$\Pi_i = \frac{1}{2}(v_i - h_i), \qquad \Pi_{ij} = \frac{1}{2}(e_i \stackrel{\sim}{\otimes} e_j + \mathrm{i}\,\epsilon_{ijk}\, e_0 \stackrel{\sim}{\otimes} e_k) \quad (i \neq j)\,, \tag{B.5}$$

where $v_i = -e_0 \otimes e_0 + e_i \otimes e_i$, and $h_i = g - v_i$. Then, $\{\Pi_i, \Pi_{jk}, \Pi_{kj}\}$, is an orthonormal frame of eigentensors of the Bel-Robinson tensor T. Moreover, T takes the canonical expression [8]:

$$T = \sum_{i=1}^{3} t_i\, \Pi_i \otimes \Pi_i + \sum_{(ijk)} \tau_i\, \Pi_{jk} \otimes \Pi_{jk} + \sum_{(ijk)} \bar{\tau}_i\, \Pi_{kj} \otimes \Pi_{kj} \,. \tag{B.6}$$

where (ijk) is an even permutation of (123).

In the more regular case I_r the scalar M is not real, and T has nine different eigenvalues, three real ones and three pairs of complex conjugate: $\{t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3\}$.

Type IM^-

In this case M is a negative real number different from -6, and T has six different eigenvalues, two real ones, a simple and a double, and two double and two simple complex conjugate eigenvalues. Then, the Bel-Robinson tensor T takes the canonical expression (B.6) with the eigenvalues restricted by $t_1 = t_2$ and $\tau_1 = \tau_2 \neq 0$ [8].

Type IM^{-6}

In this case M is the real number -6, and T has three triple eigenvalues, one real and a pair of complex conjugate. Then, the Bel-Robinson tensor T takes the canonical expression (B.6) with the eigenvalues restricted by $t_1 = t_2 = t_3$ and $\tau_1 = \tau_2 = \tau_3$ [8].

Type IM^+

In this case M is a positive real number, and T has six different real eigenvalues, three simple ones and three double ones. Then, the Bel-Robinson tensor T takes the canonical expression (B.6) with the eigenvalues restricted by $\tau_i = \bar{\tau}_i$ [8].

Type IM^∞

In this case M is infinity, and T has three different real eigenvalues with multiplicities 2, 2, 5. Then, the Bel-Robinson tensor T takes the canonical expression (B.6) with the eigenvalues restricted by $t_1 = t_2 \neq 0$, $\tau_3 = \bar{\tau}_3 = -t_1$ and $t_3 = \tau_1 = \tau_2 = 0$ [8].

Type D

Let $\{e_0, e_i\}$ be a canonical frame of a type D Weyl tensor, that is, the pairs $\{e_0, e_1\}$ and $\{e_2, e_3\}$ generate the Weyl principal 2-planes. Then, the Bel-Robinson tensor T takes the canonical expression (B.6) with the eigenvalues restricted by $t_2 = t_3 = \tau_1 \neq 0$, $t_1 = 4t_2$ and $\tau_2 = \tau_3 = -2t_2$ [8].

Type II

A type II Weyl tensor admits a double null Debever direction ℓ and two simple ones k_1, k_2. Moreover, a time-like principal plane exists which contains direction ℓ. Let k be the other null direction in the principal plane. Then, from an adapted null frame of vectors $\{\ell, k, m, \overline{m}\}$ we can define the frame of 2-tensors $\{\Pi, \Lambda, \mathrm{K}, \mathrm{N}, \overline{\mathrm{N}}, \Omega, \overline{\Omega}, \mathrm{M}, \overline{\mathrm{M}}\}$ given by:

$$\begin{aligned} &\Pi = -\tfrac{1}{2}(\ell \tilde{\otimes} k + m \tilde{\otimes} \overline{m}), \qquad && \mathrm{N} = -\tfrac{1}{\sqrt{2}}\,\ell \tilde{\otimes} \overline{m}, \qquad && \Omega = \tfrac{1}{\sqrt{2}}\,k \tilde{\otimes} m, \\ &\mathrm{M} = m \otimes m, && \Lambda = -\,\ell \otimes \ell, && \mathrm{K} = -\,k \otimes k\,. \end{aligned} \tag{B.7}$$

Then, the Bel-Robinson eigenvalues are restricted by $t_2 = t_3 = \tau_1 \neq 0$, $t_1 = 4t_2$ and $\tau_2 = \tau_3 = -2t_2$, and the Bel-Robinson tensor T takes the canonical expression [8]:

$$\begin{aligned} T &= 4\,t_2\,\Pi \otimes \Pi - 2\,t_2(\Omega \,\tilde{\otimes}\, \mathrm{N} + \overline{\Omega} \,\tilde{\otimes}\, \overline{\mathrm{N}} - \mathrm{N} \otimes \mathrm{N} - \overline{\mathrm{N}} \otimes \overline{\mathrm{N}}) \\ &+ t_2(\Lambda \,\tilde{\otimes}\, \mathrm{K} + \mathrm{M} \,\tilde{\otimes}\, \overline{\mathrm{M}} - \Lambda \,\tilde{\otimes}\, \overline{\mathrm{M}} - \Lambda \,\tilde{\otimes}\, \mathrm{M} + \Lambda \otimes \Lambda)\,. \end{aligned} \tag{B.8}$$

Type III

A type III Weyl tensor admits a triple null Debever direction ℓ and a simple one k. Then, from an adapted null frame of vectors $\{\ell, k, m, \overline{m}\}$ we can define the frame of 2-tensors $\{\Pi, \Lambda, \mathrm{K}, \mathrm{N}, \overline{\mathrm{N}}, \Omega, \overline{\Omega}, \mathrm{M}, \overline{\mathrm{M}}\}$ given in (B.7). Then, all the eigenvalues vanish, and the Bel-Robinson tensor T takes the canonical expression [8]:

$$T = \Lambda \,\tilde{\otimes}\, \Pi + \mathrm{N} \,\tilde{\otimes}\, \overline{\mathrm{N}}\,. \tag{B.9}$$

Type N

A type N Weyl tensor admits a quadruple null Debever direction ℓ. Then, all the eigenvalues vanish, and the Bel-Robinson tensor T takes the canonical expression [8]:

$$T = \ell \otimes \ell \otimes \ell \otimes \ell\,. \tag{B.10}$$

Appendix C. Some constraints on the super-energy quantities

From expressions (2) and (7) we easily obtain the following relations between the associated quadratic scalars:

$$\begin{aligned} (T,T) &= \tau^2 - 4\,|q_\perp|^2 + 6\,|t_\perp|^2 - 4\,|Q_\perp|^2 + |T_\perp|^2\,, \\ (Q,Q) &= -\tau^2 + 3\,|q_\perp|^2 - 3\,|t_\perp|^2 + |Q_\perp|^2\,, \\ (t,t) &= \tau^2 - 2\,|q_\perp|^2 + |t_\perp|^2\,, \\ (q,q) &= -\tau^2 + |q_\perp|^2\,. \end{aligned} \tag{C.1}$$

The scalars (T,T) and (Q,Q) are invariants as stated in (8). Moreover, from the Bergqvist and Lankinen conditions [13] we also obtain:

$$3\,(t,t) + 4\,(q,q) = \frac{1}{2}\alpha^2 \tag{C.2}$$

Consequently, the amount of the super-energy quantities are submitted to the
Quadratic scalar constraints

$$\begin{aligned} 4\alpha^2 &= \tau^2 - 4\,|q_\perp|^2 + 6\,|t_\perp|^2 - 4\,|Q_\perp|^2 + |T_\perp|^2\,, \\ \alpha^2 &= \tau^2 - 3\,|q_\perp|^2 + 3\,|t_\perp|^2 - |Q_\perp|^2\,, \\ \tfrac{1}{2}\alpha^2 &= -\tau^2 - 2\,|q_\perp|^2 + 3\,|t_\perp|^2\,. \end{aligned} \tag{C.3}$$

On the other hand, from the Bergqvist and Lankinen conditions [13] we obtain the following
Quadratic vectorial constraints

$$\begin{aligned} Q_\perp(t_\perp) - \tau\, q_\perp &= 0\,, \\ T_\perp(Q_\perp) - 3Q_\perp(t_\perp) + 3t_\perp(q_\perp) - \tau\, q_\perp &= 0\,. \end{aligned} \tag{C.4}$$

Quadratic 2-order tensorial constraints

$$\begin{aligned} T_\perp(t_\perp) + Q_\perp \overset{2}{\times} Q_\perp + 2Q_\perp(q_\perp) - 3\tau\, t_\perp + 2\, t_\perp \cdot t_\perp - 3q_\perp \otimes q_\perp &= 0\,, \\ Q_\perp \overset{2}{\times} Q_\perp - q_\perp \otimes q_\perp - (\tau^2 - |t_\perp|^2)\gamma &= 0\,, \\ T_\perp \overset{3}{\times} T_\perp - 3Q_\perp \overset{2}{\times} Q_\perp + 3t_\perp \cdot t_\perp - q_\perp \otimes q_\perp - \alpha^2 \gamma &= 0\,, \end{aligned} \tag{C.5}$$

Appendix D.

Proof of proposition 3

Proposition 3 states that the following relation holds:

$$(t,t) \geq \chi \geq \frac{1}{2}\alpha^2 \geq 0, \tag{D.1}$$

and moreover, it specifies when each of the three involved inequalities becomes strict or is an equality depending on the different Petrov-Bel types.

In types N and III the Bel-Robinson eigenvalues vanish. Then, expressions (4) and (6) imply $\chi = \frac{1}{2}\alpha = 0$.

In types D and II the real Bel-Robinson eigenvalues satisfy $t_2 = t_3 = \tau_1 \neq 0$, $t_1 = 4t_2$. Then, expressions (4) and (6) imply $\chi = \frac{9}{2}t_2^2 = \frac{1}{2}\alpha \neq 0$.

In type I we obtain, from the canonical form (B.6),

$$16\,\chi^2 = \Big(\sum_{i=1}^{3} t_i^2\Big)^2 = \sum_{i=1}^{3} t_i^4 + 2\sum_{i<j}^{3} t_i^2 t_j^2 = \sum_{i=1}^{3} t_i^4 + 2\sum_{k=1}^{3} |\tau_k|^4 \geq \sum_{i=1}^{3} [t_i^4 + \tau_i^4 + \bar{\tau}_i^4] = \mathrm{tr}\, T^4 = 4\alpha^4\,,$$

where the last equality has been proved in [7], $\mathrm{tr}\, T^4$ denoting the trace of the fourth power of T as an endomorphism.

From the definition (7) of t, the first inequality in (D.1) writes:

$$(t,t) = T^2(u,u,u,u) \geq \chi\,.$$

In type N, $T^2 = 0$, and then $T^2(u,u,u,u) = 0 = \chi$.

In type III, the canonical form (B.9) implies $T^2(u,u,u,u) = (l,x)^2 > 0 = \chi$.

In type II, an arbitrary observer u in terms of the Weyl canonical frame takes the expression $u = \lambda(e^{\phi}\ell + e^{-\phi}k) + \mu(e^{\mathrm{i}\sigma}m + e^{-\mathrm{i}\sigma}\bar{m})$, $2(\lambda^2 - \mu^2) = 1$. Then, from the canonical form (B.8) we obtain

$$T^2(u,u,u,u) = \chi\left[1 + 4\mu^2 + 2\mu^4 \sin^2 2\sigma + 8\left(\frac{1}{3}\lambda^2 e^{-2\phi} - \mu^2\cos 2\sigma\right)^2\right] > \chi\,.$$

Finally, we study types I and D. In [7] we have introduced a second order superenergy tensor $T_{(2)}$ associated with the traceless part $W_{(2)}$ of the square W^2 of the Weyl tensor W. That is, $T_{(2)}$ is defined as (1) by changing W by $W_{(2)}$. It follows that $T_{(2)}$ has the same properties as T [7]. Then, we can apply to it the last inequality in expression (8) of theorem 2: if e_0 is a principal observer, for any observer u we have:

$$T_{(2)}(u,u,u,u) \geq T_{(2)}(e_0,e_0,e_0,e_0)\,. \tag{D.2}$$

But, for any observer u, $T_{(2)}(u,u,u,u) = T^2(u,u,u,u) - \frac{1}{3}\alpha^2$. Thus, (D.2) holds by substituting $T_{(2)}$ by T^2, and we obtain:

$$T^2(u,u,u,u) \geq T^2(e_0,e_0,e_0,e_0) = \frac{1}{4}\sum_{i=1}^{3} t_i^2 = \chi\,. \tag{D.3}$$

Proof of proposition 4

Proposition 4 states that the following relation holds:

$$-(q,q) \geq \xi^2 \geq \frac{1}{4}\alpha^2 \geq 0, \tag{D.4}$$

and moreover, it specifies when each of the three involved inequalities becomes strict or is an equality depending on the different Petrov-Bel types.

From (6), (C.2) and (D.1) we obtain

$$-4(q,q) = 3(t,t) - \frac{1}{2}\alpha^2 \geq 3\chi - \frac{1}{2}\alpha = 4\xi^2 \geq \alpha^2\,.$$

Moreover, every inequality becomes strict (or an equality) when the corresponding inequality in (D.1) becomes.

References

[1] Ferrando J J and Sáez J A 2012 *Class. Quantum Grav.* **29** 075012
[2] Bel L 1958 *C. R. Acad. Sci.* **247** 1094
[3] Bel L 1958 *C. R. Acad. Sci.* **248** 1297
[4] Bel L 1962 *Cah. de Phys.* **16** 59. Bel L 2000 *Gen. Rel. Grav.* **32** 2047
[5] García-Parrado Gómez-Lobo A 2008 *Class. Quantum Grav.* **25** 015006
[6] McIntosh C B and Arianrhod R 1990 *Class. Quantum Grav.* **7** L213
[7] Ferrando J J and Sáez J A 2009 *Gen. Rel. Grav.* **41** 1695
[8] Ferrando J J and Sáez J A 2010 *Gen. Rel. Grav.* **42** 1469
[9] Ferrando J J and Sáez J A 1997 *Class. Quantum Grav.* **14** 129
[10] Ferrando J J and Sáez J A 2004 *Gen. Rel. Grav.* **36** 2497
[11] Bonilla M A G and Senovilla J M M 1997 *Phys. Rev. D* **78** 783
[12] Rainich G Y 1925 *Trans. Am. Math. Soc.* **27** 106
[13] Bergqvist G, Lankinen P 2004 *Class. Quantum Grav.* **21** 3499
[14] Lichnerowicz A 1960 *Ann. Mat. Pura Appl.* **50** 1
[15] Senovilla J M M 2000 *Class. Quantum Grav.* **17**, 2799
[16] Senovilla J M M 2001 *Class. Quantum Grav.* **18**, 5299
[17] Kinnerrsley W 1969 *J. Math. Phys.* **10** 1195
[18] Stephani E, Kramer H, McCallum M A H, Hoenselaers C and Hertl E 2003 *Exact Solutions of Einstein's Field Equations* (Cambridge Univ. Press, Cambridge)
[19] Szekeres P 1965 *J. Math. Phys.* **9** 1387
[20] Brans C H 1975 *J. Math. Phys.* **16** 1008
[21] Kasner A (1921) *Am. J. Phys.* **43** 217
[22] Ferrando J J and Sáez J A 2003 *Class. Quantum Grav.* **20** 5291
[23] Ferrando J J and Sáez J A 2002 *Class. Quantum Grav.* **19** 2437
[24] Petrov A Z 1962 Gravitational field geometry as the geometry of automorphisms it Recent developments in General Relativity (Oxford: Pergamon) p 379
[25] McIntosh C B G 1992 *Gen. Rel. Grav.* **24** 757
[26] Taub A H 1951 *Ann. Math.* **53** 472